Choice functions as a tool to model uncertainty

Arthur Van Camp

5 March 2019

Imprecise probabilities

=

broaden probability theory in order to deal with imprecision and indecision.

Imprecise probabilities

$=$

broaden probability theory in order to deal with imprecision and indecision.

Basic idea of imprecise probabilities: decisions and choice.

(Precise) choice functions

A preference relation is $\prec$ is a strict weak order (irreflexive and transitive, with transitive absence of preference binary relation) on a set of options.

(Precise) choice functions

A preference relation is $\prec$ is a strict weak order (irreflexive and transitive, with transitive absence of preference binary relation) on a set of options.

The goal is to observe a subject's choices to learn about her preferences.

(Precise) choice functions

A preference relation is $\prec$ is a strict weak order (irreflexive and transitive, with transitive absence of preference binary relation) on a set of options.

The goal is to observe a subject's choices to learn about her preferences.

A: set of option (non-empty but finite)

$C(A)$: chosen or admissible or non-rejected options

$R(A)$: rejected options ($R(A) = A \setminus C(A)$)

(Precise) choice functions

A preference relation is $\prec$ is a strict weak order (irreflexive and transitive, with transitive absence of preference binary relation) on a set of options.

The goal is to observe a subject's choices to learn about her preferences.

A: set of option (non-empty but finite)

$C(A)$: chosen or admissible or non-rejected options

$R(A)$: rejected options ($R(A) = A \setminus C(A)$)

A choice function C is a map

$$C\colon \mathscr{Q} \to \mathscr{Q} \cup \{\emptyset\} \colon A \mapsto C(A) \text{ such that } C(A) \subseteq A.$$

Induced choice functions

Given a preference relation $\prec$, the induced choice function $C_{\prec}$ is defined by

$$C_{\prec}(A) := \{f \in A : (\forall g \in A) f \not\prec g\}.$$

Induced choice functions

Given a preference relation $\prec$, the induced choice function $C_{\prec}$ is defined by

$$C_{\prec}(A) := \{f \in A : (\forall g \in A) f \not\prec g\}.$$

A choice function C is rationalisable if there is a preference relation $\prec$ such that $C = C_{\prec}$; if this is the case, then C is rationalised by $\prec$.

Induced choice functions

Given a preference relation $\prec$, the induced choice function $C_\prec$ is defined by

$$C_\prec(A) := \{f \in A : (\forall g \in A) f \nprec g\}.$$

A choice function C is rationalisable if there is a preference relation $\prec$ such that $C = C_\prec$; if this is the case, then C is rationalised by $\prec$.

Proposition: if C is rationalised by $\prec$, then $\prec$ can be retrieved by

$$f \prec g \Leftrightarrow (\exists A \in \mathscr{Q}) f \in C(A) \text{ and } g \in R(A).$$

Induced choice functions

Given a preference relation $\prec$, the induced choice function $C_{\prec}$ is defined by

$$C_{\prec}(A) := \{f \in A : (\forall g \in A) f \not\prec g\}.$$

When is C rationalisable?

Induced choice functions

Given a preference relation $\prec$, the induced choice function $C_\prec$ is defined by

$$C_\prec(A) := \{f \in A : (\forall g \in A) f \not\prec g\}.$$

When is C rationalisable?

Non-emptiness

$$C(A) \neq \emptyset.$$

Houthakker's axiom

If $f, g \in A_1 \cap A_2$, $f \in C(A_1)$ and $g \in C(A_2)$, then $f \in C(A_2)$.

Induced choice functions

Given a preference relation $\prec$, the induced choice function $C_\prec$ is defined by

$$C_\prec(A) \coloneqq \{f \in A : (\forall g \in A) f \not\prec g\}.$$

When is C rationalisable?

Non-emptiness

$$C(A) \neq \emptyset.$$

Houthakker's axiom

If $f, g \in A_1 \cap A_2$, $f \in C(A_1)$ and $g \in C(A_2)$, then $f \in C(A_2)$.

Proposition: C is non-empty and satisfies Houthakker's Axiom if and only if C is rationalisable.

Example: probabilities

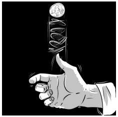
A random variable X takes values in the finite possibility space $\mathscr{X}$.

We have a probability mass function p on $\mathscr{X}$.

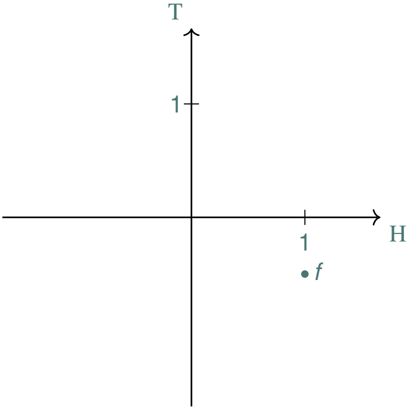
What choice function describes my beliefs?

What we choose between: gambles

A gamble $f\colon \mathscr{X} \to \mathbb{R}$ is an uncertain reward whose value is $f(X)$, and we collect all gambles in $\mathscr{L} = \mathbb{R}^{\mathscr{X}}$.

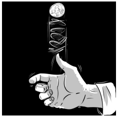


$\mathscr{X} = \{\mathrm{H}, \mathrm{T}\}$

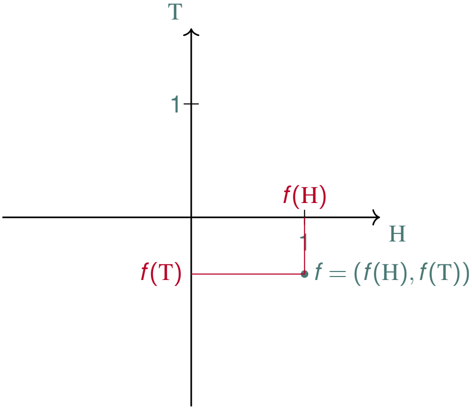


What we choose between: gambles

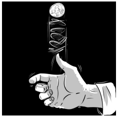
A gamble $f\colon \mathscr{X} \to \mathbb{R}$ is an uncertain reward whose value is $f(X)$, and we collect all gambles in $\mathscr{L} = \mathbb{R}^{\mathscr{X}}$.



$\mathscr{X} = \{\mathrm{H}, \mathrm{T}\}$

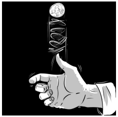


Fair coin



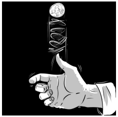
Assessment: “The coin is fair.”

Fair coin



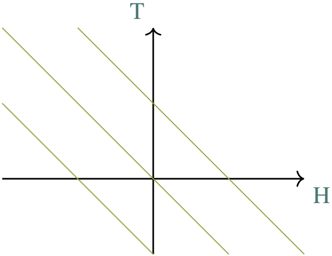
Assessment: “The coin is fair.”

Fair coin

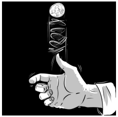


Assessment: “The coin is fair.”

$$f <_p g \Leftrightarrow E_p(f) < E_p(g)$$

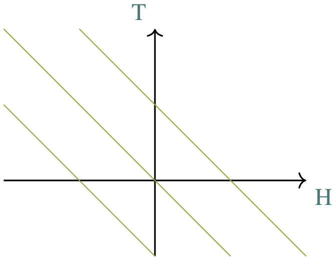


Fair coin



Assessment: "The coin is fair."

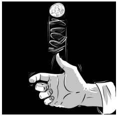
$$f <_p g \Leftrightarrow E_p(f) < E_p(g)$$



$$C_p(A) = \{f \in A : (\forall g \in A) E_p(g) \leq E_p(f)\} = \arg\max\{f \in A : E_p(f)\}$$

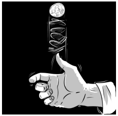
C_p is non-empty and satisfies Houthakker's Axiom, and is therefore rationalisable, with $\prec = <_p$.

Sets of probabilities



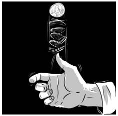
Assessment: "H is at least as likely as T."

Sets of probabilities



Assessment: "H is at least as likely as T."

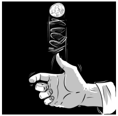
Sets of probabilities



Assessment: "H is at least as likely as T."

$$C_{\mathscr{M}}(A) = \bigcup_{p \in \mathscr{M}} C_p(A) = \{f \in A : (\exists p \in \mathscr{M}) f \in C_p(A)\}.$$

Sets of probabilities



Assessment: "H is at least as likely as T."

$$C_{\mathscr{M}}(A) = \bigcup_{p \in \mathscr{M}} C_p(A) = \{f \in A : (\exists p \in \mathscr{M}) f \in C_p(A)\}.$$

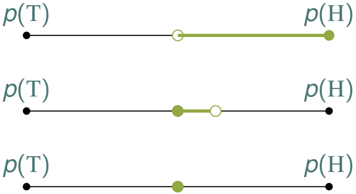
$C_{\mathscr{M}}$ does not satisfies Houthakker's Axiom, and is therefore not rationalisable.

“Non-Archimedean” beliefs

Assessment: “The coin is infinitesimally biased towards H, but not by any definite amount.”

“Non-Archimedean” beliefs

Assessment: “The coin is infinitesimally biased towards H, but not by any definite amount.”



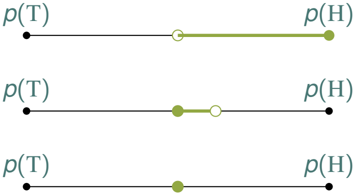
“H is more likely than T.”

still biased by some amount

“H is equally likely as T.”

“Non-Archimedean” beliefs

Assessment: “The coin is infinitesimally biased towards H, but not by any definite amount.”



“H is more likely than T.”

still biased by some amount

“H is equally likely as T.”

(Sets of) probabilities cannot capture this belief.

Binary choice: Sets of desirable gambles

Preference relation $\succ$ on $\mathscr{L}$. For all f, g, h in $\mathscr{L}$ and real $\lambda > 0$:

$$f \succ g \Leftrightarrow \lambda f + h \succ \lambda g + h.$$

Binary choice: Sets of desirable gambles

Preference relation $\succ$ on $\mathscr{L}$. For all f, g, h in $\mathscr{L}$ and real $\lambda > 0$:

$$f \succ g \Leftrightarrow \lambda f + h \succ \lambda g + h.$$

The subject's set of desirable gambles: which gambles does she strictly prefer to 0?

How does that work?

Binary choice: Sets of desirable gambles

Preference relation $\succ$ on $\mathscr{L}$. For all f, g, h in $\mathscr{L}$ and real $\lambda > 0$:

$$f \succ g \Leftrightarrow \lambda f + h \succ \lambda g + h.$$

The subject's set of desirable gambles: which gambles does she strictly prefer to 0?

How does that work?

$$f \succ g \Leftrightarrow f - g \succ 0 \Leftrightarrow f - g \in D$$

for all f and g in $\mathscr{L}$.

Binary choice: Sets of desirable gambles

Preference relation $\succ$ on $\mathscr{L}$. For all f, g, h in $\mathscr{L}$ and real $\lambda > 0$:

$$f \succ g \Leftrightarrow \lambda f + h \succ \lambda g + h.$$

The subject's set of desirable gambles: which gambles does she strictly prefer to 0?

How does that work?

$$f \succ g \Leftrightarrow f - g \succ 0 \Leftrightarrow f - g \in D$$

for all f and g in $\mathscr{L}$.

To summarise:

$$D = \{f \in \mathscr{L} : f \succ 0\}.$$

Binary choice: Sets of desirable gambles

Set of desirable gambles $D = \{f \in \mathscr{L} : f \succ 0\} \subseteq \mathscr{L}$

Working with sets of desirable gambles is simple and elegant.

They include lower previsions and sets of probabilities as a special case.

They generalise conservative logical inference (natural extension).

Coherent sets of desirable gambles

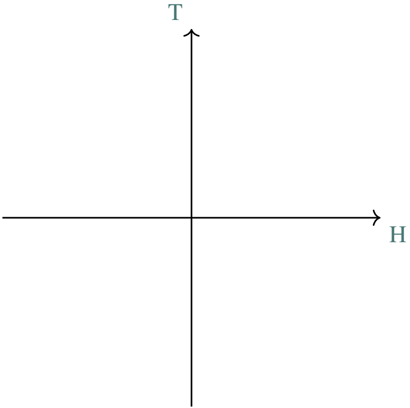
A set of desirable gambles D is called coherent if for all f, g in $\mathscr{L}$:

D1. if $f \leq 0$ then $f \notin D$ [not desiring non-positivity]

D2. if $f > 0$ then $f \in D$ [accepting partial gains]

D3. if $f \in D$ then $\lambda f \in D$ for all real $\lambda > 0$ [scaling]

D4. if $f, g \in D$ then $f + g \in D$ [addition]



Coherent sets of desirable gambles

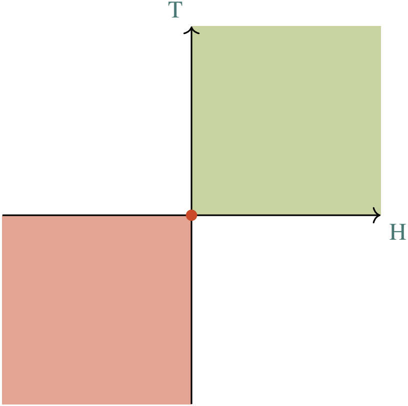
A set of desirable gambles D is called coherent if for all f, g in $\mathscr{L}$:

D1. if $f \leq 0$ then $f \notin D$ [not desiring non-positivity]

D2. if $f > 0$ then $f \in D$ [accepting partial gains]

D3. if $f \in D$ then $\lambda f \in D$ for all real $\lambda > 0$ [scaling]

D4. if $f, g \in D$ then $f + g \in D$ [addition]



Coherent sets of desirable gambles

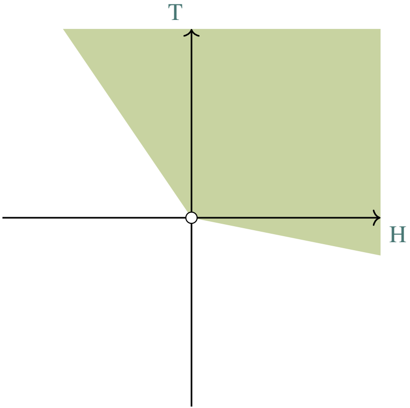
A set of desirable gambles D is called coherent if for all f, g in $\mathscr{L}$:

D1. if $f \leq 0$ then $f \notin D$ [not desiring non-positivity]

D2. if $f > 0$ then $f \in D$ [accepting partial gains]

D3. if $f \in D$ then $\lambda f \in D$ for all real $\lambda > 0$ [scaling]

D4. if $f, g \in D$ then $f + g \in D$ [addition]



Coherent sets of desirable gambles

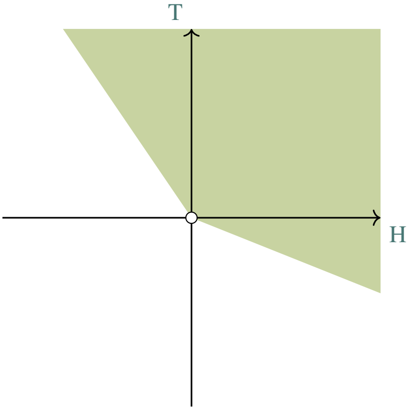
A set of desirable gambles D is called coherent if for all f, g in $\mathscr{L}$:

D1. if $f \leq 0$ then $f \notin D$ [not desiring non-positivity]

D2. if $f > 0$ then $f \in D$ [accepting partial gains]

D3. if $f \in D$ then $\lambda f \in D$ for all real $\lambda > 0$ [scaling]

D4. if $f, g \in D$ then $f + g \in D$ [addition]



Coherent sets of desirable gambles

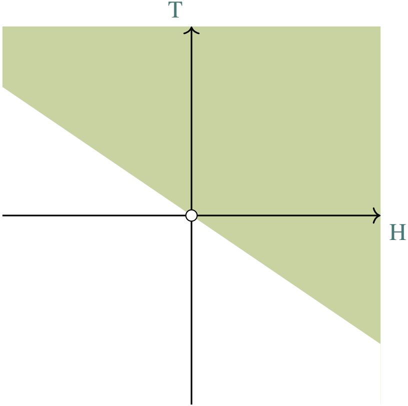
A set of desirable gambles D is called coherent if for all f, g in $\mathscr{L}$:

D1. if $f \leq 0$ then $f \notin D$ [not desiring non-positivity]

D2. if $f > 0$ then $f \in D$ [accepting partial gains]

D3. if $f \in D$ then $\lambda f \in D$ for all real $\lambda > 0$ [scaling]

D4. if $f, g \in D$ then $f + g \in D$ [addition]



halfspace: precise probability model

Coherent sets of desirable gambles

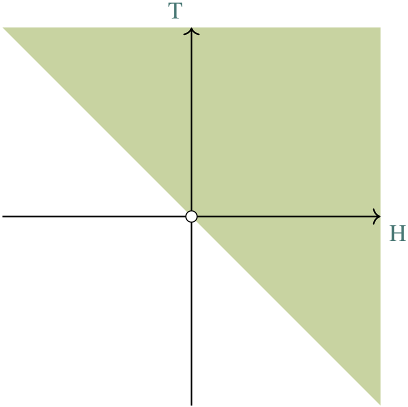
A set of desirable gambles D is called coherent if for all f, g in $\mathscr{L}$:

D1. if $f \leq 0$ then $f \notin D$ [not desiring non-positivity]

D2. if $f > 0$ then $f \in D$ [accepting partial gains]

D3. if $f \in D$ then $\lambda f \in D$ for all real $\lambda > 0$ [scaling]

D4. if $f, g \in D$ then $f + g \in D$ [addition]



halfspace: precise probability model

Coherent sets of desirable gambles

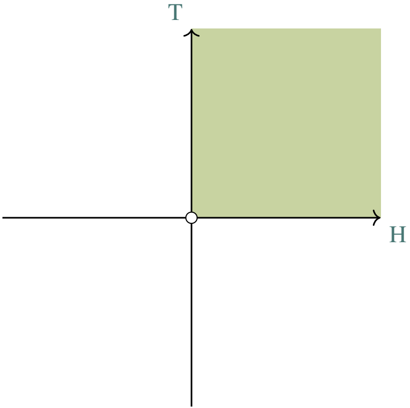
A set of desirable gambles D is called coherent if for all f, g in $\mathscr{L}$:

D1. if $f \leq 0$ then $f \notin D$ [not desiring non-positivity]

D2. if $f > 0$ then $f \in D$ [accepting partial gains]

D3. if $f \in D$ then $\lambda f \in D$ for all real $\lambda > 0$ [scaling]

D4. if $f, g \in D$ then $f + g \in D$ [addition]



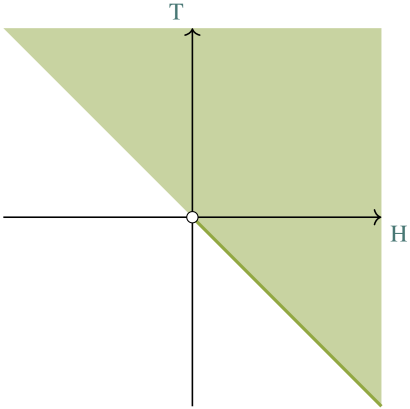
smallest coherent D_{v}: the vacuous model

Example: Choice based on desirability

Assessment: "The coin is infinitesimally biased towards H, but not by any definite amount."

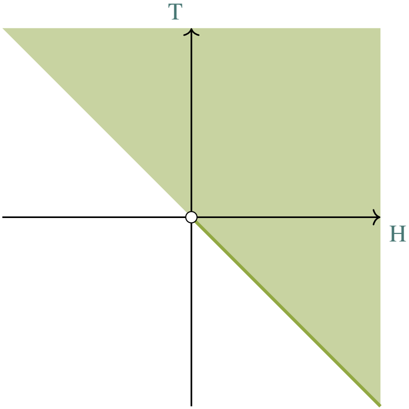
Example: Choice based on desirability

Assessment: “The coin is infinitesimally biased towards H,
but not by any definite amount.”



Example: Choice based on desirability

Assessment: "The coin is infinitesimally biased towards H, but not by any definite amount."



$C_D(A) = \{f \in A : (\forall g \in A) g - f \notin D\}$, so $f \in C_D(A) \Leftrightarrow A - \{f\} \cap D = \emptyset$.

Disjunctive statements

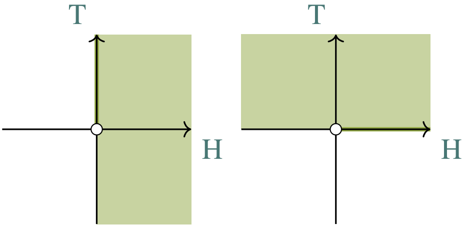
Assessment: “The coin has two identical sides of unknown type.”

Disjunctive statements

Assessment: “The coin has two identical sides of unknown type.”

Consider the coherent sets of desirable gambles

$$D_{\mathrm{H}} = \{f \in \mathscr{L} : f(\mathrm{H}) > 0\} \cup \mathscr{L}_{>0} \text{ and } D_{\mathrm{T}} = \{f \in \mathscr{L} : f(\mathrm{T}) > 0\} \cup \mathscr{L}_{>0}.$$



Then $D = D_{\mathrm{H}} \cap D_{\mathrm{T}} = \mathscr{L}_{>0}$ is the vacuous set of desirable gambles.

Coherent choice functions

We call a choice function C on $\mathscr{L}$ coherent if for all A, A_1, A_2 in $\mathscr{Q}$, f, g in $\mathscr{L}$ and λ in $\mathbb{R}_{>0}$:

- C1. $C(A) \neq \emptyset$; [non-emptiness]
- C2. if $f < g$ then $f \in R(\{f, g\})$; [non-triviality]
- C3a. if $A \subseteq R(A_1)$ and $A_1 \subseteq A_2$ then $A \subseteq R(A_2)$; [Sen's condition α]
- C3b. if $A_1 \subseteq R(A_2)$ and $A \subseteq A_1$
  then $A_1 \setminus A \subseteq R(A_2 \setminus A)$; [Aizerman's condition]
- C4a. if $A_1 \subseteq C(A_2)$ then $\lambda A_1 \subseteq C(\lambda A_2)$; [scaling]
- C4b. if $A_1 \subseteq C(A_2)$ then $A_1 + \{f\} \subseteq C(A_2 + \{f\})$. [addition]

First axiomatisation: Seidenfeld, Schervish and Kadane, 2010.

Reasoning with choice functions

C_1 is not more informative than C_2 if $C_1(A) \supseteq C_2(A)$ for all A.

Reasoning with choice functions

C_1 is not more informative than C_2 if $C_1(A) \supseteq C_2(A)$ for all A.

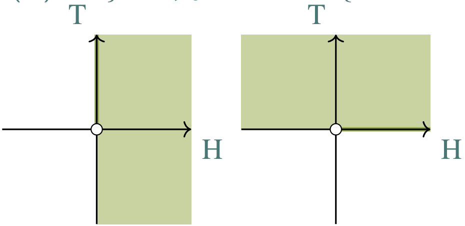
The smallest coherent choice function is

$$C_{\mathrm{v}}(A) = \max(A) = \{f \in A : (\forall g \in A) f \nless g\} = C_{D_{\mathrm{v}}}(A).$$

Coin with identical sides

Assessment: "The coin has two identical sides of unknown type."

$D_{\mathrm{H}} = \{f \in \mathscr{L} : f(\mathrm{H}) > 0\} \cup \mathscr{L}_{>0}$ and $D_{\mathrm{T}} = \{f \in \mathscr{L} : f(\mathrm{T}) > 0\} \cup \mathscr{L}_{>0}$.

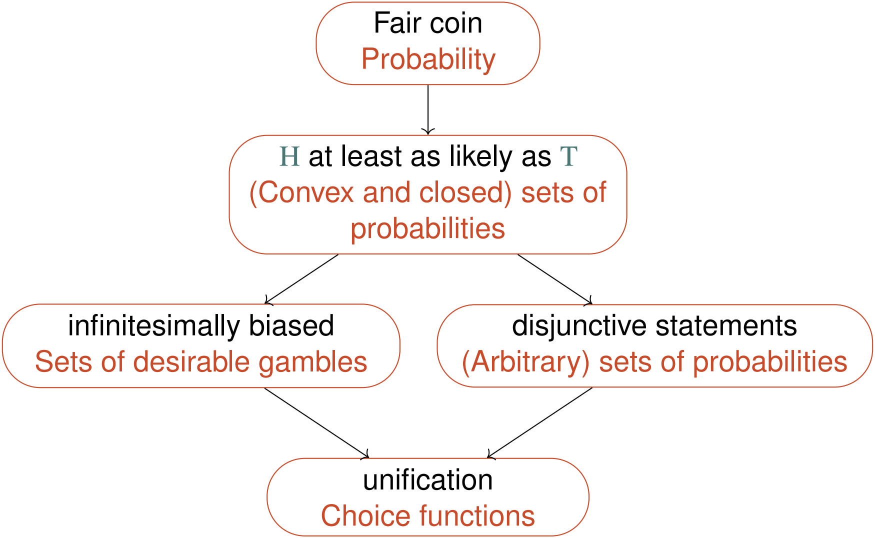


$$C_{\mathrm{H}}(A) = \{f \in A : (\forall g \in A) g - f \notin D_{\mathrm{H}}\} = \arg\max\{f \in \max(A) : f(\mathrm{H})\}$$
$$C_{\mathrm{T}}(A) = \arg\max\{f \in \max(A) : f(\mathrm{T})\}$$

But

$$C(A) = \arg\max\{f \in \max(A) : f(\mathrm{H})\} \cup \arg\max\{f \in \max(A) : f(\mathrm{T})\}.$$

Overview


Fair coin
Probability
H at least as likely as T
(Convex and closed) sets of probabilities
infinitesimally biased
Sets of desirable gambles
disjunctive statements
(Arbitrary) sets of probabilities
unification
Choice functions