General framework for preference modeling

Meltem Ozturk

Lamsade, Universite Paris Dauphine
ozturk@lamsade.dauphine.fr
http://www.lamsade.dauphine.fr/∼ozturk

Les articles de la presentation

- M. Ozturk, Coherence conditions for preference modeling with ordered points, *Journal of Mathematical Psychology*, Volume 79, 2017, pages 44-52 p.
- M. Ozturk , M. Pirlot et A. Tsoukias, Representing preferences using intervals, *Artificial Intelligence Journal*, 175, pages 1194-1222, 2011

Preorder

Definition ((Total) Preorder)

A binary relation $R = \langle P, I \rangle$ on A, is a preorder iff there exists g on A such that $\forall x, y \in A$,

$$\begin{cases} xPy & \iff & g(x) > g(y) \\ xIy & \iff & g(x) = g(y) \end{cases}$$

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Preferences and Numbers

Comparison

Given objects having a numerical representation how do these compare (before, after, near, better, worst, similar)?

Representation

Given a binary relation among objects what is a suitable numerical representation for it?

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Evaluation:

	a	b	c
Evaluation	25	11	9

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D_1	a	b	c
a	I	P	P
b	P^{-1}	I	P
c	P^{-1}	P^{-1}	I

D_2	a	b	c
a	I	P	P
b	P^{-1}	I	I
c	P^{-1}	I	I

Preferences and Numbers

Decision maker preferences:

	a	b	c
a	I	I	P
b	I	I	I
c	P^{-1}	I	I

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a	b	c
[4,7]	[1,5]	[0,2]

Two “different” approaches

Preference and Indifference

Use only two binary relations, an asymmetric one (preference) and a symmetric one (indifference). The symmetric relation can always been seen as the union of the identity relation, I_o and two inverse asymmetric relations. We call that a $\langle P, I \rangle$ preference structure.

More preference relations

Use n ($n > 2$) asymmetric relations and the identity relation I_o. This amounts getting an indifference relation I and $n - 1$ preference relations. A well known case are the $\langle P, Q, I \rangle$ preference structures.

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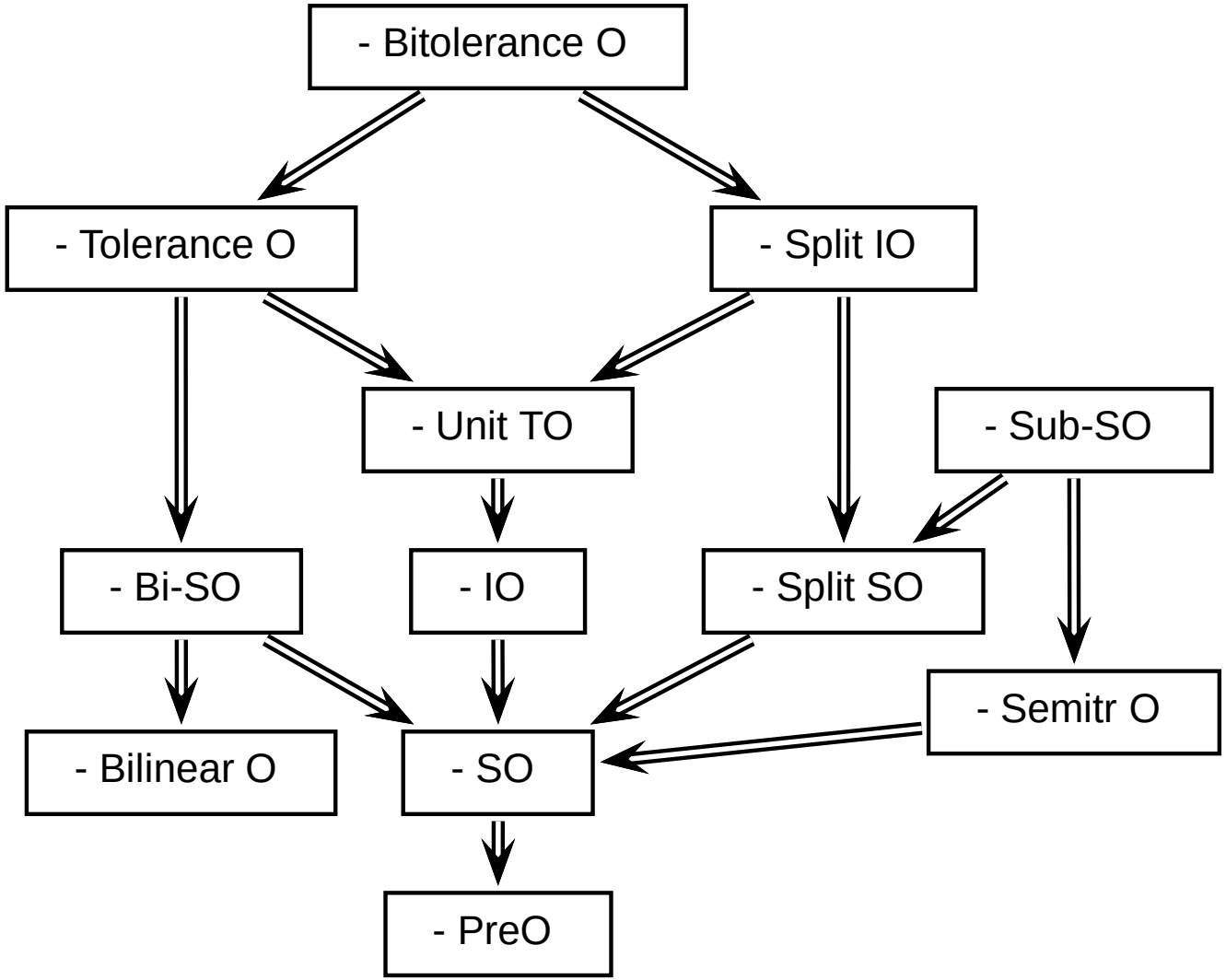
Fishburn's classification



intervals, ordered points?

Let a, b, c 3 alternatives on A with
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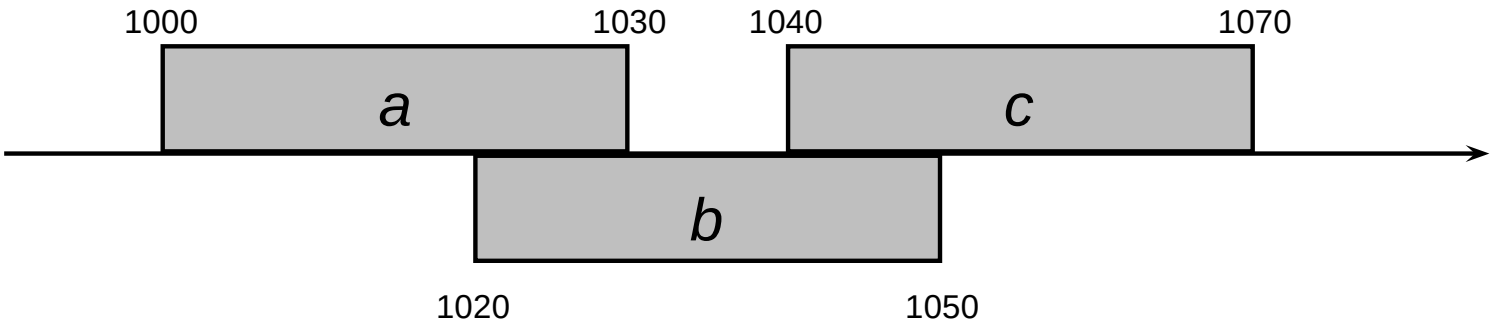

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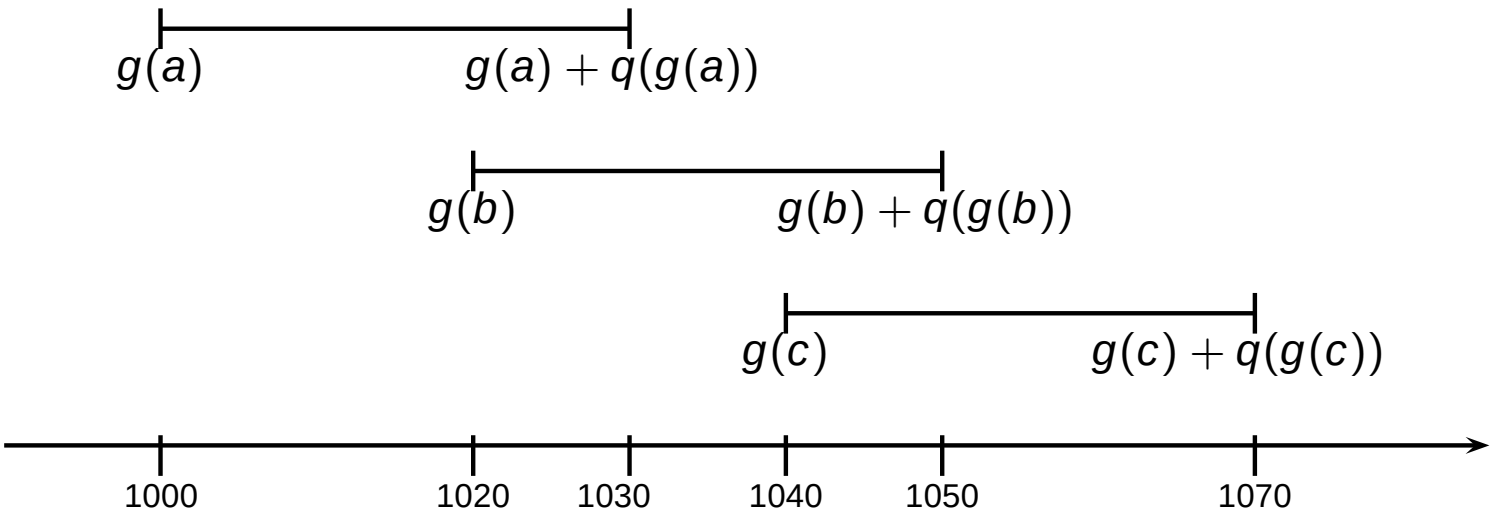


Thresholds: aIb and bIc but cPa

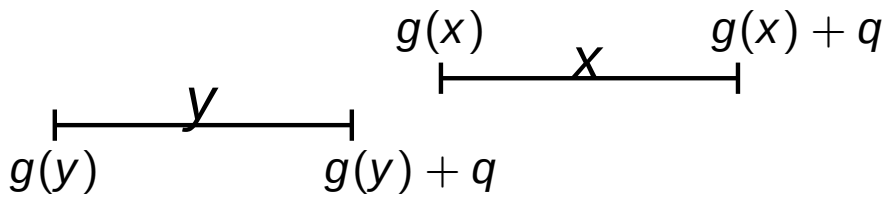
intervals, ordered points?

intransitive: Luce ([Luce 1956]), quasiorders.

Definition (SemiOrder)

A reflexive relation $R = \langle P, I \rangle$ on A, is a semi order iff there exists g on A and a non negative constant q such that $\forall x, y \in A$,

$$\begin{cases} xPy & \iff & g(x) > g(y) + q, \\ xIy & \iff & |g(x) - g(y)| \leq q. \end{cases}$$

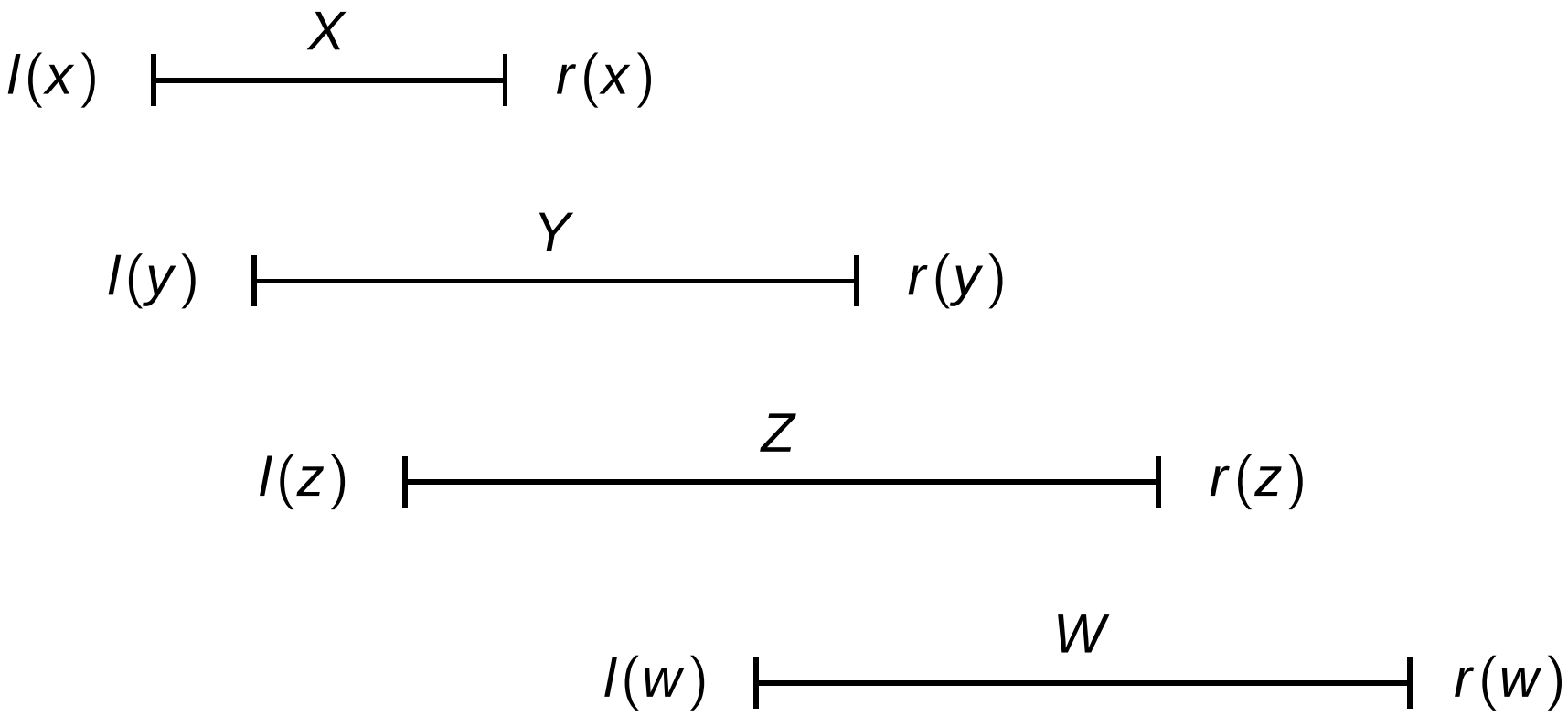


xPy

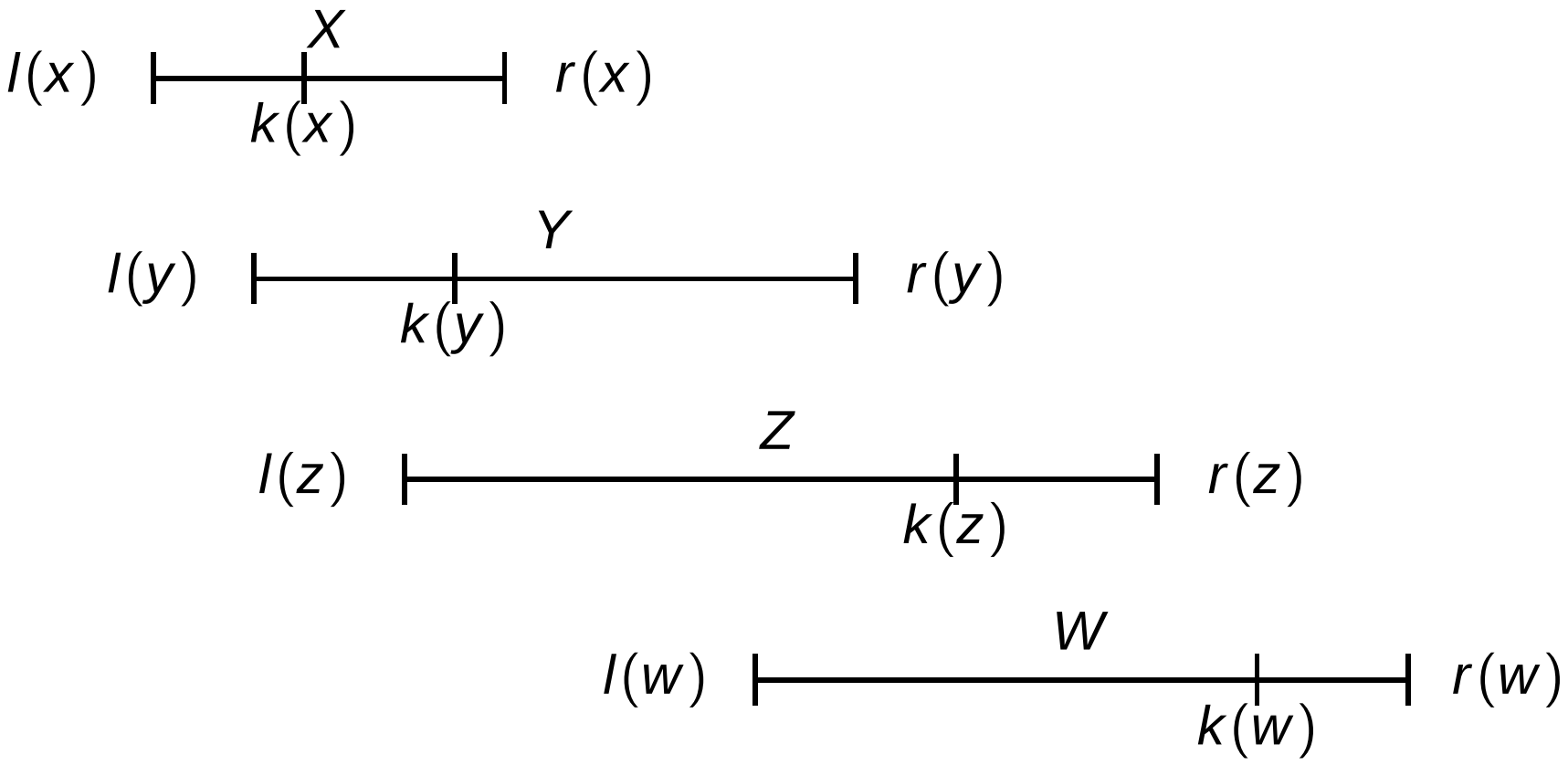
Why intervals?

- Imprecision in measurement and/or information (length 10cm$\pm$2mm).
- Uncertain information (price between 10 € and 12 €).
- Uncertain assessments (quality between average and good).
- Positive and negative reasons in evaluation.

An example: 2 points

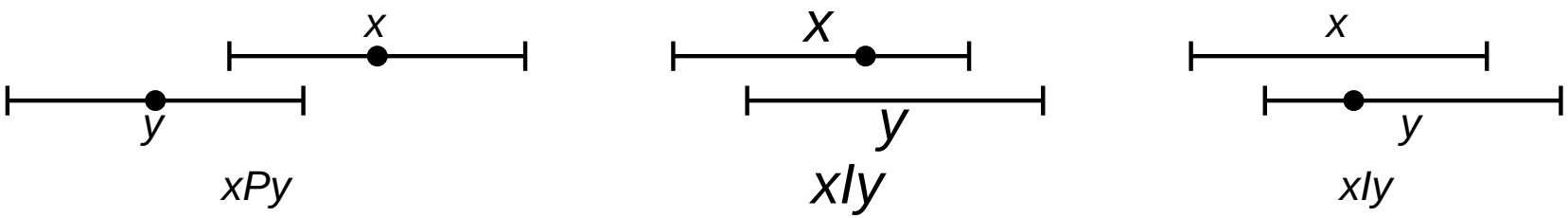


An example: 3 points

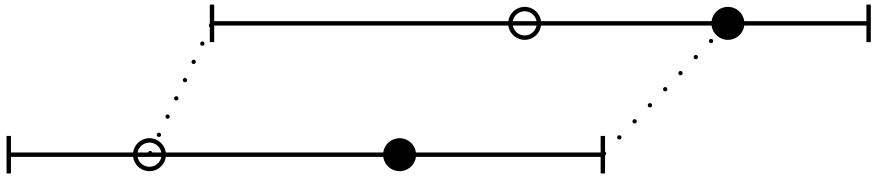


Examples with ordered points

- Split semiorder



- Bitolerance order



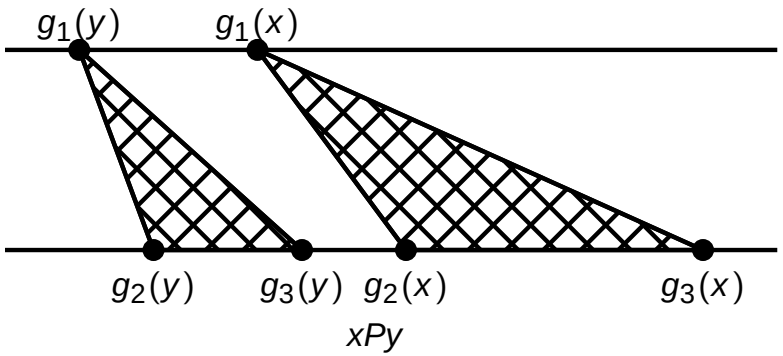
xPy

Example : Triangle orders

Definition

$P \cup I$ is a triangle order if it is defined as the intersection of one weak order and one interval order.

$$\begin{cases} xPy \iff \begin{cases} g_1(x) > g_1(y), \\ g_2(x) > g_3(y), \end{cases} \\ \forall x,\ \forall i \in \{1,2\},\ g_{i+1}(x) \geq g_i(x). \end{cases}$$



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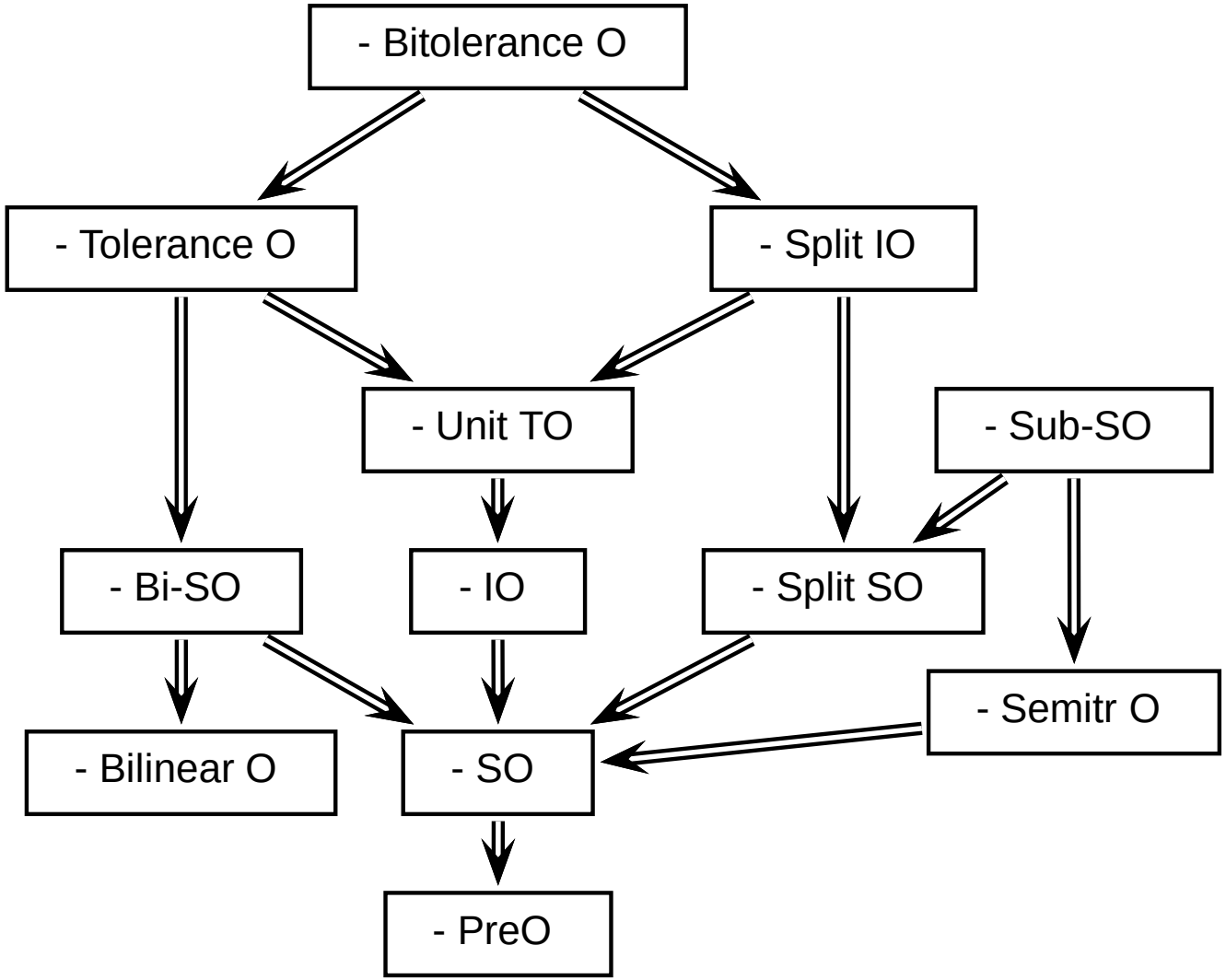


Problems

1. Is this classification exhaustive?
2. There is no unique characterisation of these structures. Some are characterised using forbidden posets, some characterising the binary relations, some through their numerical representation.
3. There is no general framework for such structures

Characterization

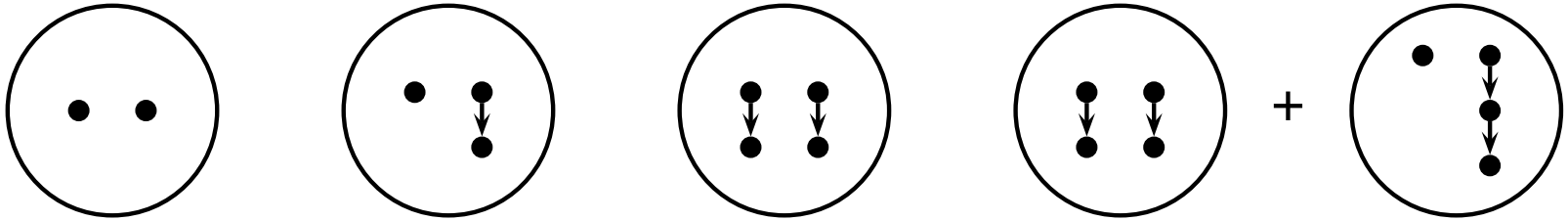
- *Representative mode*: interval order

$$xPy \iff g(x) > g(y) + q(g(y))$$

- *Relational mode*: interval order

$$\begin{cases} P.I.P \subset P, \\ P \cup I \text{ is reflexive and complete} \end{cases}$$

- *Forbidden mode*



total order	preorder	interval order	semiorder
(1+1)	(1+2)	(2+2)	(2+2) and (1+3)

What are we looking for?

Intervals comparison

A general framework under which objects represented by n points of the reals can be compared.

Representation Theorems

Necessary and sufficient conditions for which a numerical representation (using intervals) fits a certain binary relation.

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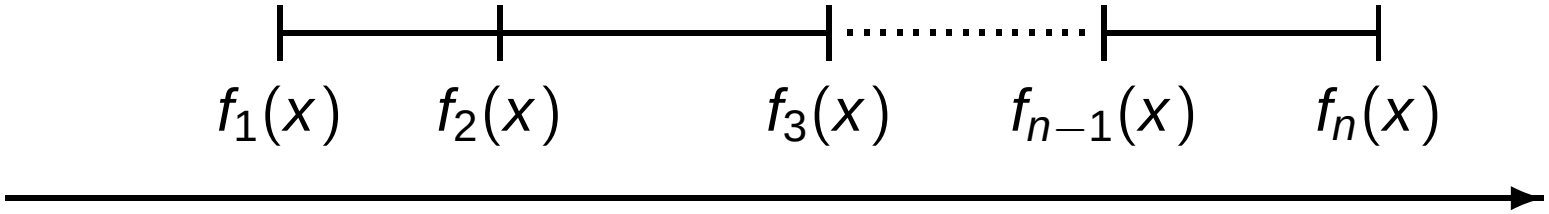
n-point intervals

n-point interval : n ordered points, $f_1(x), f_2(x), \ldots f_n(x)$, such that for all $x \in A$ and all i in $\{1, \ldots, n-1\}$, $f_i(x) \leq f_{i+1}(x)$.



Why *n* ordered points?

- When the evaluation of an object has many possibilities such as the price of an object in n shops.
- When the evaluation of an object is a fuzzy number
- When some special points are used as thresholds in order to define preference intensities.
- In the context of decision under complete uncertainty where there is no knowledge about the probabilities
- When a lottery has a uniform probability distribution with n possibilities

Why n ordered points?

Aim: analyse “all the reasonable” comparison rules (axiomatization) giving rise to a complete preference structure composed by P (strict preference) and I (indifference, the complement of P)

Relative Positions

Definition (Relative position)

A relative position $\varphi(x,y)$ *is the n-tuple* $\langle \varphi_1(x,y), \ldots, \varphi_n(x,y) \rangle$ *where* $\varphi_i(x,y)$ *represents the number of* j *such that* $f_i(x) \leq f_j(y)$

$\varphi(x,y) = (1,0,0) \qquad \varphi^T(x,y) = \varphi(y,x) = (3,3,2)$

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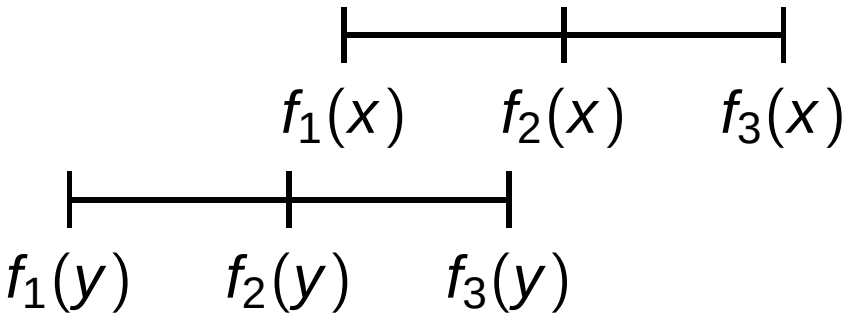

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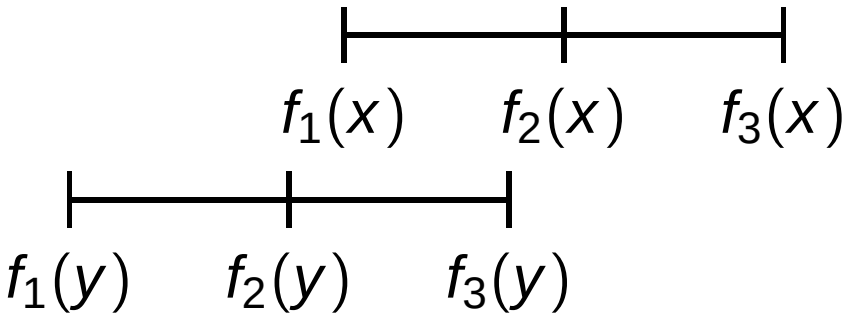

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Intervals of Time

Allen introduced 13 relations

7 of them are “basic”, the others are the inverse:

Name	***Not.***	***Position***	$\varphi(x,y)$	$\varphi(y,x)$
Equal	$x = y$	$f_1(x) = f_1(y) \wedge f_2(x) = f_2(y)$	$(2,1)$	$(2,1)$
Before	$x\underline{b}y$	$f_1(x) > f_2(y)$	$(0,0)$	$(2,2)$
Overlap	$x\underline{o}y$	$f_1(x) > f_1(y) \wedge f_2(y) > f_1(x) \wedge f_2(x) > f_2(y)$	$(1,0)$	$(2,1)$
Meets	$x\underline{m}y$	$f_1(x) = f_2(y)$	$(1,0)$	$(2,2)$
During	$x\underline{d}y$	$f_1(y) > f_1(x) \wedge f_2(x) > f_2(y)$	$(2,0)$	$(1,1)$
Starts	$x\underline{s}y$	$f_1(x) = f_1(y) \wedge f_2(x) > f_2(y)$	$(2,0)$	$(2,1)$
Finishes	$x\underline{f}y$	$f_1(y) > f_1(x) \wedge f_2(x) = f_2(y)$	$(2,1)$	$(1,1)$

How many are they?

Proposition

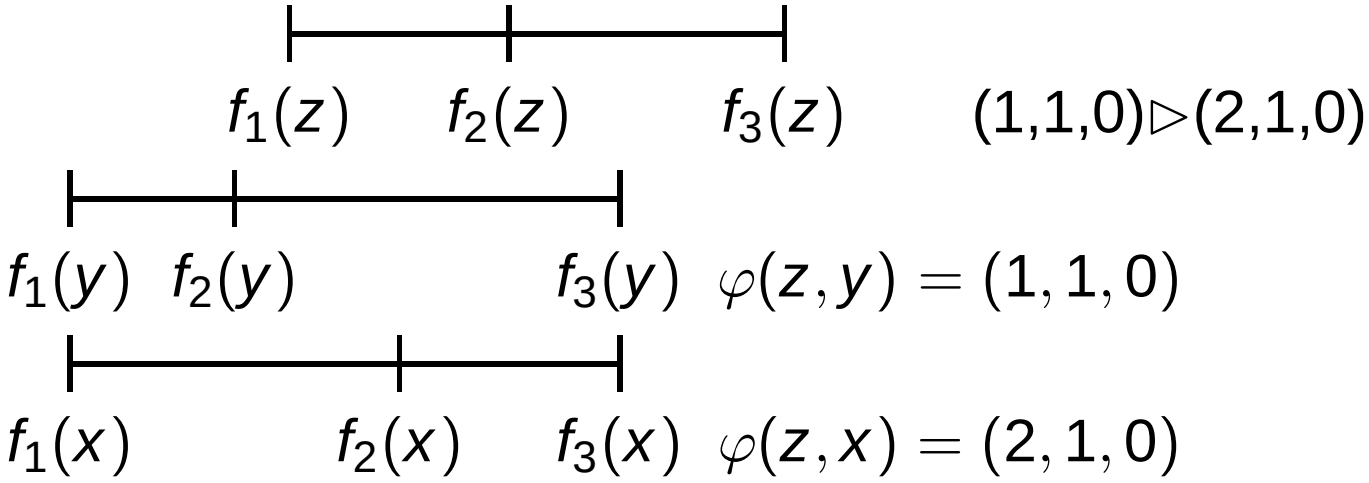
Let x and y be two n-point intervals then the number of possible relative positions $\varphi(x, y)$ is $m = \frac{(2n)!}{(n!)^2}$.

	$n = 2$	$n = 3$	$n = 4$	n
Relative positions	6	20	70	$\frac{(2n)!}{(n!)^2}$

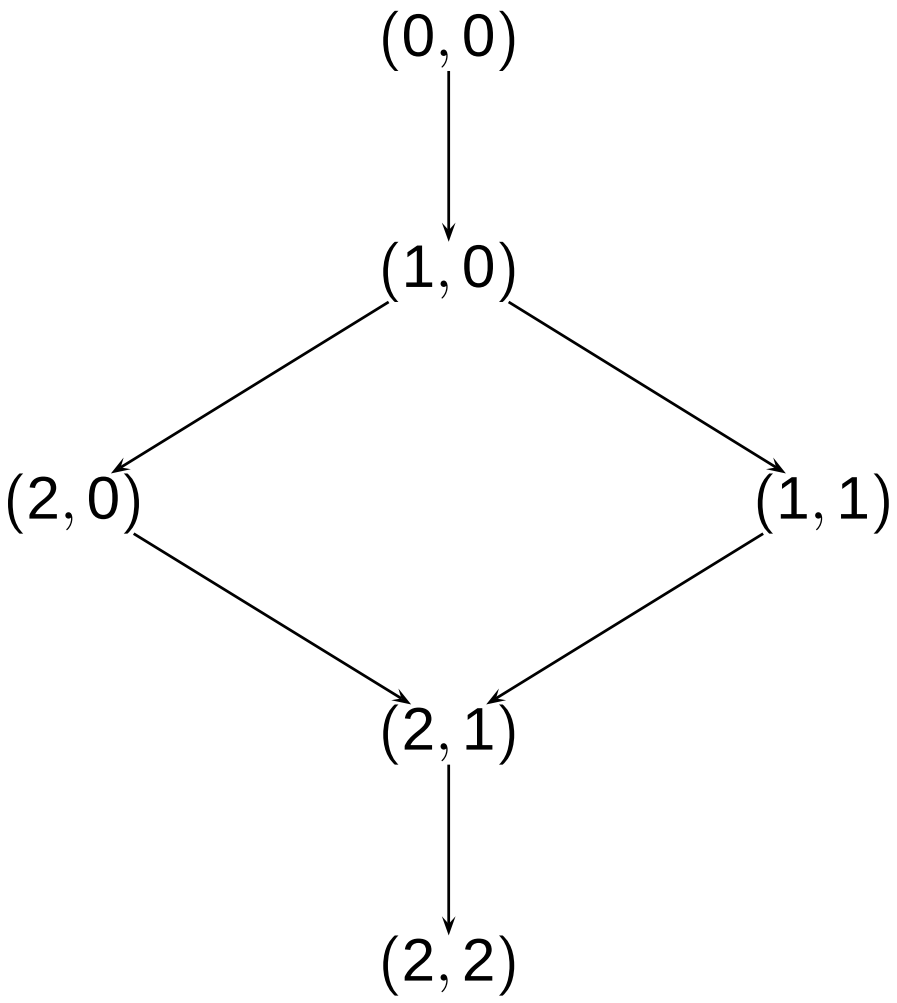
“Stronger than” Relation: $\triangleright$

Definition (“Stronger than” relation)

Let φ and φ' be two relative positions, then we say that φ is “stronger than” φ' and note $\varphi \triangleright \varphi'$ if $\forall i \in \{1, \ldots, n\},\ \varphi_i \leq \varphi'_i$.



The graph of $\triangleright$



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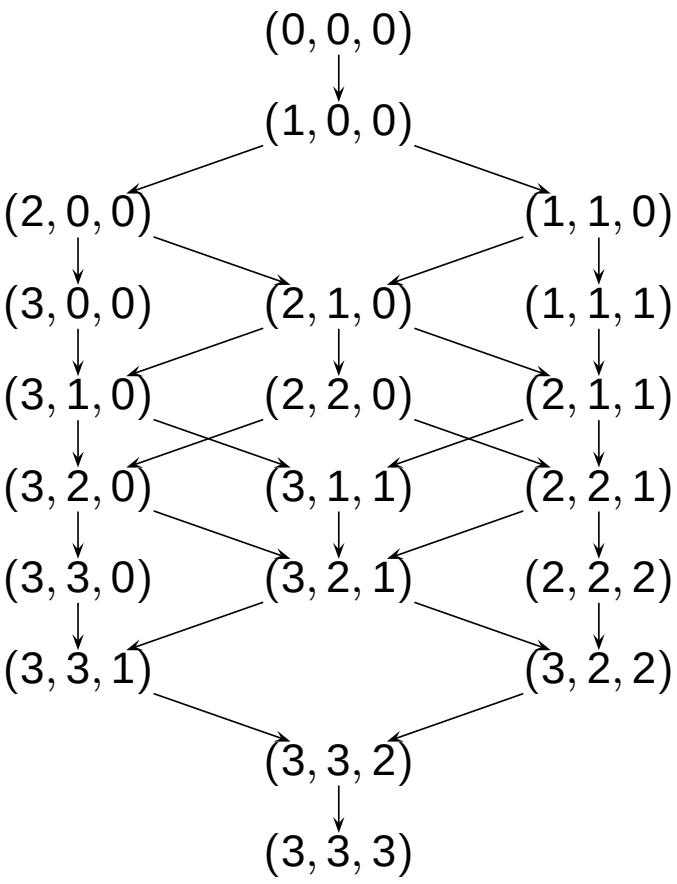

Axioms

We are interested in a complete preference relation $P \cup I$

(P is asymmetric, I is reflexive and symmetric and $P \cup I$ is complete).

- For all x, y, z, t, if $\varphi(x, y) = \varphi(z, t)$, then $P(x, y) \Longleftrightarrow P(z, t)$.
- For all x, y, z, t, if $\varphi(x, y) \rhd \varphi(z, t)$ and $P(z, t)$ then $P(x, y)$.
- The set of relative positions forming P has one and only one weakest relative position :
  For all x, y, z, t,
  if $[P(x, y)$ and $P(z, t)$ with $not(\varphi(x, y) \rhd \varphi(z, t))$ and $not(\varphi(z, t) \rhd \varphi(x, y))]$
  then $\exists \varphi(r, m)$ such that $\varphi(x, y) \rhd \varphi(r, m)$ and $\varphi(z, t) \rhd \varphi(r, m)$ and $P(r, m)$.

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How many such sets of relative positions?

Proposition

Let m be the number of sets of relative positions satisfying axioms 1-5 then

$$m = \frac{(2n)!}{(n!)^2} - \frac{1}{n+1}\binom{2n}{n}$$

	$n = 2$	$n = 3$	$n = 4$	n
Set of relative positions	4	15	56	$\frac{(2n)!}{(n!)^2} - \frac{1}{n+1}\binom{2n}{n}$

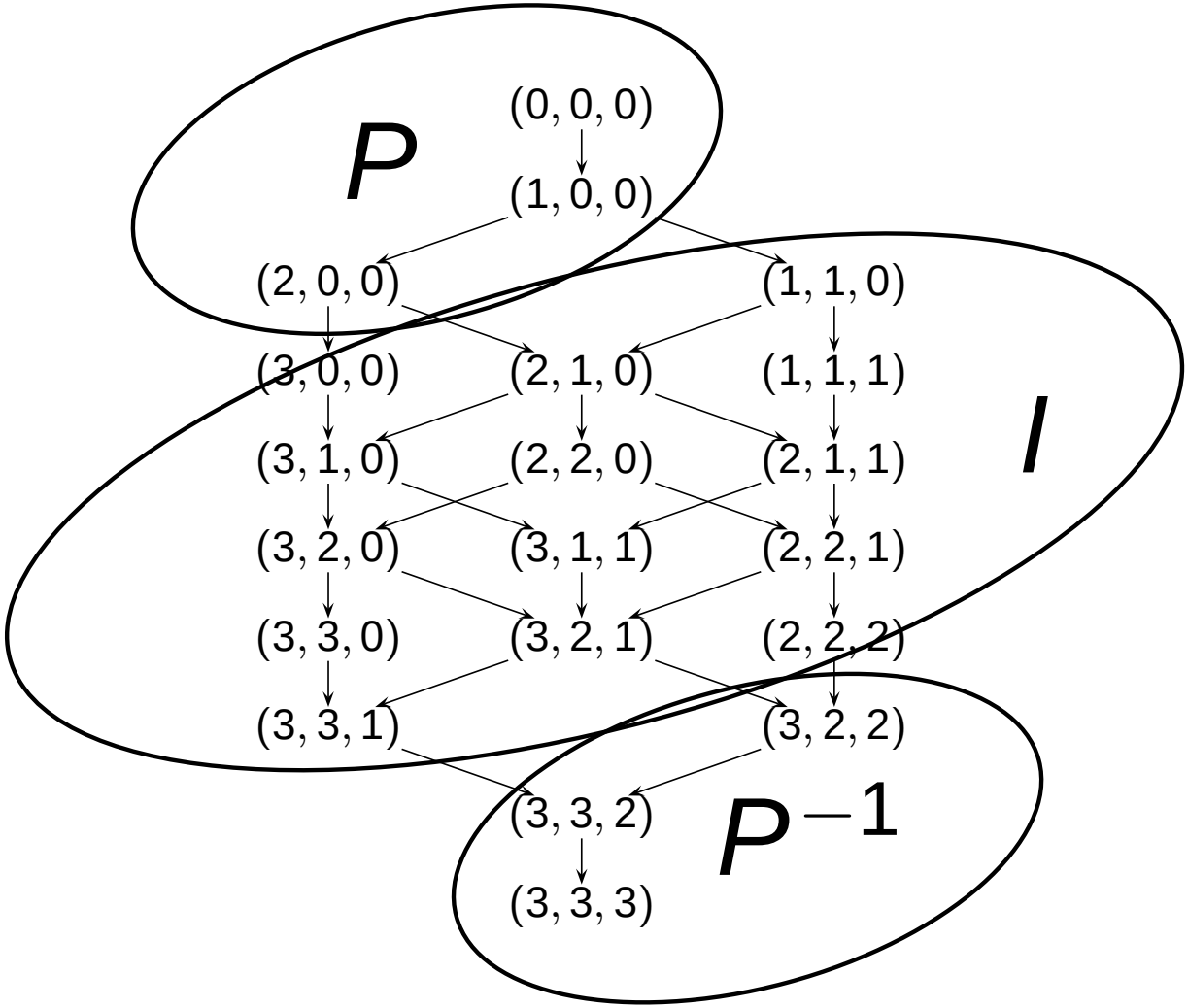
Preference Structure

Definition

Let φ be an n-tuple and x and y 2 n-point intervals. Relations $P_{\leq\varphi}$ and $I_{\leq\varphi}$ where $(n, n-1, n-2, \ldots, 1) \not\rhd \varphi$ are defined as

$$\begin{aligned} P_{\leq\varphi}(x,y) &\iff \varphi(x,y) \rhd \varphi, \\ I_{\leq\varphi}(x,y) &\iff \neg P_{\leq\varphi}(x,y) \wedge \neg P_{\leq\varphi}(y,x). \end{aligned}$$

$P_{\leq(2,0,0)}$


P
I
P^{-1}
(0, 0, 0)
(1, 0, 0)
(2, 0, 0)
(1, 1, 0)
(3, 0, 0)
(2, 1, 0)
(1, 1, 1)
(3, 1, 0)
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(3, 2, 1)
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(3, 3, 1)
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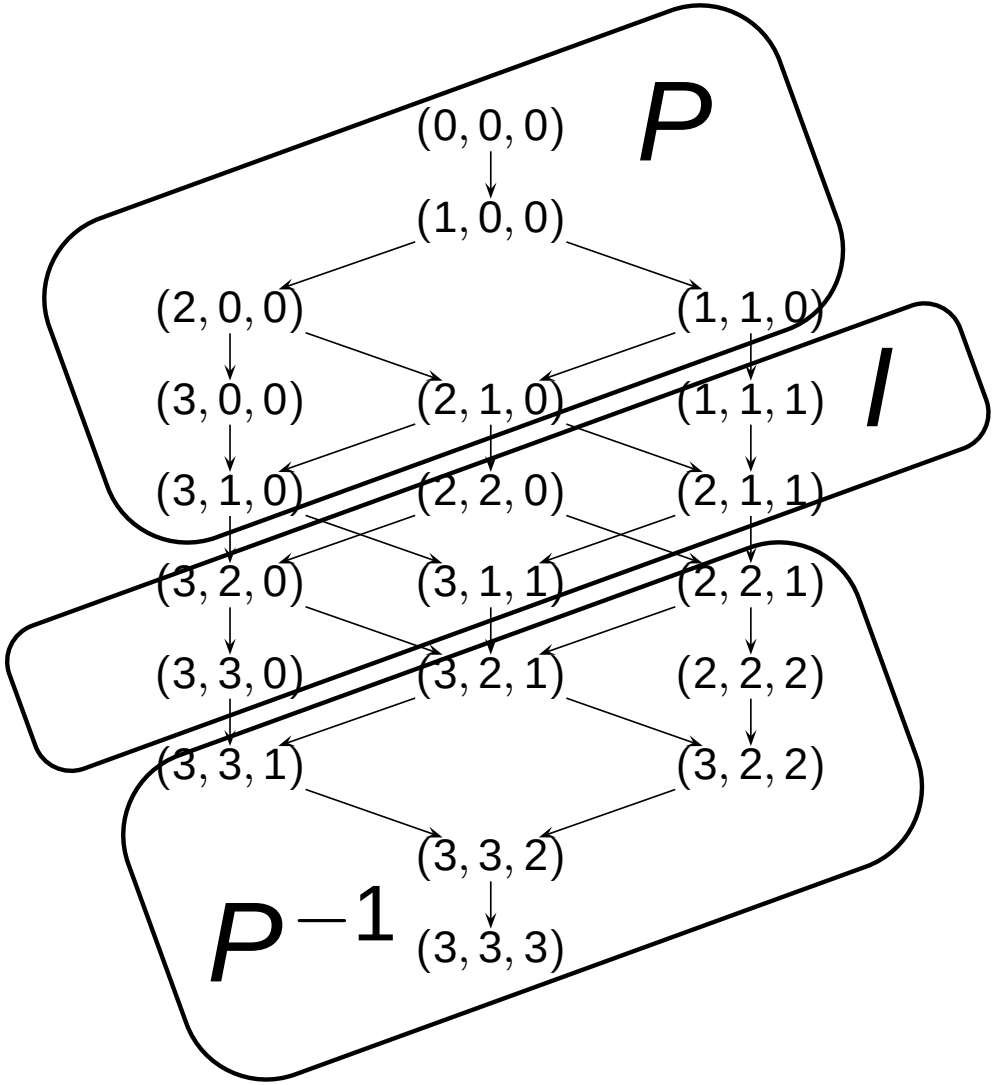
(3, 3, 2)
(3, 3, 3)

$P_{\leq(3,1,0)}$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(2, 2, 2)
(3, 3, 1)
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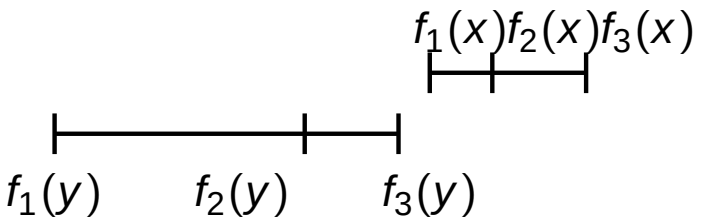
Component set

Definition

The component set $Cp_{\leq\varphi}$ is the set of couples $(n - \varphi_i, i)$ such that $\varphi_i \neq n$ and there is no $i' < i$ with $\varphi_{i'} = \varphi_i$

$P_{\leq(2,0,0)} : (0, 0, 0)$



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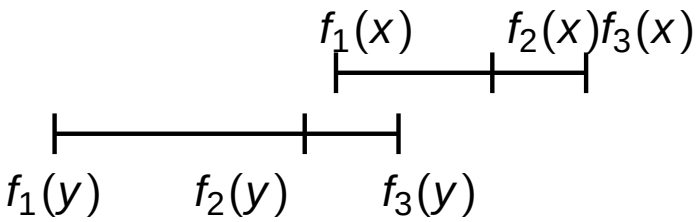

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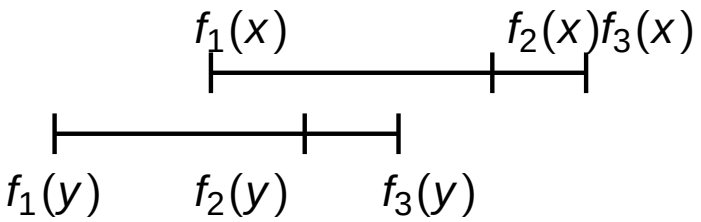
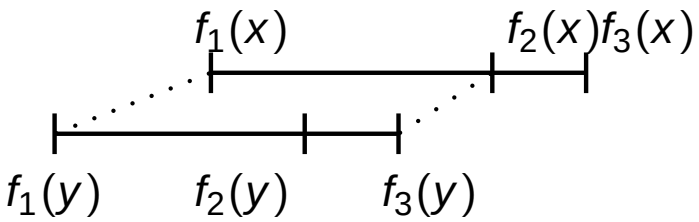


Component set

Definition

The component set $Cp_{\leq\varphi}$ is the set of couples $(n-\varphi_i, i)$ such that $\varphi_i \neq n$ and there is no $i' < i$ with $\varphi_{i'} = \varphi_i$

$$P_{\leq(2,0,0)} : (0,0,0) \cup (1,0,0) \cup (2,0,0)$$



$$Cp_{\leq(2,0,0)} = \{(1,1),(3,2)\}$$

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The component set $Cp_{\leq\varphi}$ is the set of couples $(n - \varphi_i, i)$ such that $\varphi_i \neq n$ and there is no $i' < i$ with $\varphi_{i'} = \varphi_i$

strict preference relation:

$$\forall x, y,\ P_{\leq\varphi}(x, y) \Longleftrightarrow \forall (i, j) \in Cp_{\leq\varphi},\ f_i(y) < f_j(x)$$

Some characterisations

	$\lvert Cp_{\leq\varphi}\rvert$	$(i,j) \in Cp_{\leq\varphi}$
$P_{\leq\varphi}$ is *transitive*		$\forall(i,j),\ i \geq j,$
$I_{\leq\varphi}$ is *transitive*	1	$Cp_{\leq\varphi} = \{(i,i)\}$
$P_{\leq\varphi} \cup I_{\leq\varphi}$ is a *weak oder*	1	$Cp_{\leq\varphi} = \{(i,i)\}$
$P_{\leq\varphi} \cup I_{\leq\varphi}$ is a *d-weak order*	d	$\forall(i,j),\ i = j$
$P_{\leq\varphi} \cup I_{\leq\varphi}$ is an *interval order*	1	
$P_{\leq\varphi} \cup I_{\leq\varphi}$ is a *"bi-tolerance order"*	2	$\forall(i,j),\ i \geq j,$
$P_{\leq\varphi} \cup I_{\leq\varphi}$ is a *triangle order*	2	$Cp_{\leq\varphi} = \{(l,l),(i,j)\}$ where $i \geq j$

How many representations?

	$n = 2$	$n = 3$	$n = 4$	n
weak order	2	3	4	n
d-weak order	$\binom{2}{d}$	$\binom{3}{d}$	$\binom{4}{d}$	$\binom{n}{d}$
bi-weak order	1	3	6	$\frac{n(n-1)}{2}$
3-weak order	0	1	4	$\binom{n}{3}$
interval order	1	3	6	$\frac{n(n-1)}{2}$
bitolerance order	0	0	6	
triangle order	0	2	8	

Results for 3-point intervals

Preference Structure	$\langle P_{\leq_\varphi}, I_{\leq_\varphi} \rangle$ *interval representation*
Weak Orders	$Cp_{\leq(3,3,0)} = \{(3,3)\}$ $Cp_{\leq(3,1,1)} = \{(2,2)\}$ $Cp_{\leq(2,2,2)} = \{(1,1)\}$
Bi-weak Orders	$Cp_{\leq(3,1,0)} = \{(2,2),(3,3)\}$ $Cp_{\leq(2,1,1)} = \{(1,1),(2,2)\}$ $Cp_{\leq(2,2,0)} = \{(1,1),(3,3)\}$
Three-Weak Orders	$Cp_{\leq(2,1,0)} = \{(1,1),(2,2),(3,3)\}$

Results for 3-point intervals

Preference Structure	$\langle P_{\leq_\varphi}, I_{\leq_\varphi} \rangle$ *interval representation*
Interval Orders	$Cp_{\leq(0,0,0)} = \{(3,1)\}$ $Cp_{\leq(3,0,0)} = \{(3,2)\}$ $Cp_{\leq(1,1,1)} = \{(2,1)\}$
Split Interval Orders	$Cp_{\leq(1,0,0)} = \{(3,2),(2,1)\}$
Triangle Orders	$Cp_{\leq(1,1,0)} = \{(2,1),(3,3)\}$ $Cp_{\leq(2,0,0)} = \{(1,1),(3,2)\}$
Intransitive Orders	$Cp_{\leq(3,2,0)} = \{(3,3),(1,2)\}$ $Cp_{\leq(2,2,1)} = \{(1,1),(2,3)\}$

Coherence conditions of semi orders

Definition (Semi Order)

A binary relation P is a semiorder iff P is an interval order and $\forall x, f_1(x) > f_1(y) \Longrightarrow f_2(x) > f_2(y)$

Proposition

(see [PirlotVincke97]) The following assertions are equivalent:

i. *P is a semiorder.*
ii. *P is an interval order and there exists an equivalent 2-point interval representation such that* $\exists c, \forall x, f_2(x) = f_1(x) + c$.
iii. *P is an interval order and there exists an equivalent 2-point interval representation such that* $\forall x, y$ *there is no inclusion between their intervals.*

Coherence conditions with n points

Remark: three ways to define a coherence condition with 2 ordered points are no more equivalent with more than 2 points!!
$\Longrightarrow$ Two different coherence conditions: Monotonicity and regularity

Coherence conditions with n points

Definition (Monotonicity condition)

An n-point interval representation is monotone iff $\forall x, y, \nexists i, j, x_i \subset y_j$.

Proposition

If an n-point interval representation is monotone then there exists an equivalent n-point interval representation such that $\exists c > 0, \forall x, \forall i \parallel x_i \parallel = c$.

Proposition

Let P_{φ_M} *a preference relation in a monotone representation, then* P_{φ_M} *is either a semiorder (if the first couple of* $Cp_{\leq\varphi}$ *is in form of* (i, j) *with* $i \neq j$*) or a weak order (if the first couple of* $Cp_{\leq\varphi}$ *is in form of* (i, i)*).*

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Coherence conditions with n points

Definition (Regularity condition)

An n-point interval representation is regular iff $\forall x, y \in A$ $(\nexists j, x_j \subset y_j)$ *(equivalent to* $\forall x, y \in A$, $f_1(x) > f_1(y) \Longrightarrow \forall i, f_i(x) > f_i(y)$*).*

Proposition

There exist n-point interval representations which are regular but for which there does not exist an equivalent n-point interval representation such that $\forall i, \exists c_i, \forall x, ||x_i|| = c_i$.

Proposition

Let P_{φ_R} *a preference relation defined on a set of objects having a regular representation, then* P_{φ_R} *is transitive.*

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Coherence conditions with 3 points

φ	P_φ no condition	P_φ *monotone*	P_φ *regular*
$(0,0,0)$	interval o.	semi o.	semi o.
$(1,0,0)$	split IO	semi o.	split SO
$(2,0,0)$	triangle o.	-	semi o.
$(1,1,0)$	triangle o.	-	semi o.
$(2,1,0)$	3-weak o.	weak o.	weak o.

Table: Preference structures with 3-point interval representation and coherence conditions

Coherence conditions with 4 points

φ	P_φ no condition	P_φ *monotone*	P_φ *regular*
$(0,0,0,0)$	interval o.	semi o.	semi o.
$(1,0,0,0)$	bitolerance o.	semi o.	bitolerance o.
$(2,0,0,0)$	split IO	-	split IO
$(3,0,0,0)$	triangle o.	-	semi o.
$(2,1,0,0)$	3-interval o.	semi o.	3-interval o.
$(1,1,1,0)$	triangle o.	-	semi o.
$(3,1,0,0)$	weak o.$\cap$ split IO.	-	split IO
$(2,2,0,0)$	bitolerance o.	-	bitolerance o.
$(2,1,1,0)$	split IO. $\cap$ weak o.	-	split IO
$(3,2,0,0)$	2-weak o. $\cap$ interval o.	-	semi o.
$(3,1,1,0)$	2-weak o. $\cap$ interval o.	-	semi o.
$(2,2,1,0)$	2-weak o. $\cap$ interval o.	-	semi o.
$(3,2,1,0)$	4-weak o.	weak o.	weak o.

Table: Preference structures with 4-point interval representation

Conclusion for preference structures

- Proposition of general framework: commun language for preference structures
- An axiomatic approach close to decision maker
- A systematic approach : a listing of preference structures that a Decision maker can check when he has to chose a comparison rule
- Some new preference structures, some generalization of mathematical properties (Fishburn conjecture).

Future works?

- More flexible axioms???
- Algorithmics issues about preference structures: detecting a preference structure, elicitation of a special preference structure, minimal repsentation of a preference structure...